

# Root Systems and Boundary Bootstrap

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## Abstract

The principle of boundary bootstrap plays a significant role in the algebraic study of the purely elastic boundary reflection matrix  $K_a(\theta)$  for integrable quantum field theory defined on a space-time with a boundary. However, general structure of that principle in the form as was originally introduced by Fring and Köberle has remained unclear. In terms of a new matrix  $J_a(\theta) = \sqrt{K_a(\theta)/K_{\bar{a}}(i\pi + \theta)}$ , the boundary bootstrap takes a simple form. Incidentally, a hypothesised expression of the boundary reflection matrix for simply-laced  $ADE$  affine Toda field theory defined on a half line with the Neumann boundary condition is obtained in terms of geometrical quantities of root systems à la Dorey.

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# 1. Introduction

The boundary reflection matrix in quantum field theory is conceived to describe a reflection process of particles against a (spatial) boundary of a space-time. When the property of integrability of the quantum field theory defined on a whole line continues to hold even in the presence of a boundary, consistency requirements such as the principle of boundary bootstrap strongly constrain the exact boundary reflection matrix.

A significant step toward the algebraic study of the exact boundary reflection matrix was put forward by introducing the boundary Yang-Baxter equation[1] about a decade ago. However, in case of non-diagonal reflection, this equation alone is not sufficient to allow one to find the scalar function of the boundary reflection matrix. About ten years had passed until the scalar function was finally found by introducing the crossing-unitarity relation[2].

In case of diagonal reflection where types of particles do not change, the boundary Yang-Baxter equation becomes trivially satisfied. So one need another condition like the boundary bootstrap equation[3]. Subsequently, a variety of solutions of the algebraic equations for affine Toda field theory(ATFT) has been constructed explicitly[2, 3, 4, 5, 6]. However, proper interpretations to these solutions in the framework of Lagrangian quantum field theory was not given<sup>‡</sup>.

In order to have a direct access to the boundary reflection matrix, perturbative approach has been developed and a quantum field theoretic definition of the boundary reflection matrix was proposed[9]. A complete set of conjectures for the exact boundary reflection matrix for simply-laced *ADE* affine Toda field theory defined on a half line with the Neumann boundary condition was obtained[10]. It is noted that each of the solutions is not ‘minimal’ among all possible solutions of the algebraic equations in the usual sense of the total number of poles and zeros on the physical strip.

General structure of the boundary bootstrap in the form as was originally introduced by Fring and Köberle has remained unclear[3]. In this letter, a new matrix  $J_a(\theta) = \sqrt{K_a(\theta)/K_a(i\pi + \theta)}$  which renders the boundary bootstrap more tractable is introduced and a hypothesised expression of the boundary reflection matrix for simply-laced *ADE* affine Toda field theory defined on a half line with the Neumann boundary condition is obtained in terms of geometrical quantities of root systems à la Dorey.

In section 2, the geometry associated with the Coxeter element of the Weyl group of root systems which also played an important role in the geometric formulation of the

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<sup>‡</sup>There are some works which aim to relate parameters appearing in the boundary potential and formal parameters arising from solutions of the algebraic equations; for the sine-Gordon theory at a generic point in semi-classical analysis[7] and at the free fermion point[8] where one may use the method[2] of mode expansion for the field as an operator.

$S$ -matrix[11, 12] is briefly reviewed to set up a notation. In section 3, the new form of the boundary bootstrap is introduced in terms of  $J_a(\theta)$  function and some discussions on its properties are given. Then, the hypothesised expression of the boundary reflection matrix for simply-laced  $ADE$  affine Toda field theory defined on a half line with the Neumann boundary condition is obtained in terms of geometrical quantities of root systems. Finally some conclusions are made in section 4.

## 2. Geometry of root systems

The geometric expression of the exact  $S$ -matrix for simply-laced  $ADE$  affine Toda field theory can be written in various ways depending on a chosen set of representatives of the Weyl orbits. The notation of ref.[13] will be taken here.

Let simple roots  $\alpha_i$  ( $i = 1, \dots, n$ ) for a simply-laced Lie algebra  $g$  with rank  $n$  be divided into two sets such that the roots in each set are mutually orthogonal. The members of the two sets are distinguished by assigning a colour to them, either black or white. Let the simple roots be labelled so that

$$\bullet = \{1, 2, \dots, k\}, \quad \circ = \{k+1, k+2, \dots, n\}. \quad (2.1)$$

For any root  $x$ , an elementary Weyl reflection  $w_i$  corresponding to the simple root  $\alpha_i$  is defined by

$$w_i(x) = x - 2 \frac{x \cdot \alpha_i}{\alpha_i^2} \alpha_i. \quad (2.2)$$

The Weyl group is generated by these elementary Weyl reflections. A Coxeter element of the Weyl group is a product over the simple roots of the elementary Weyl reflections in any choice of ordering. With the above choice of ordering of the simple roots, the Coxeter element is defined by

$$w = w_\bullet w_\circ = w_1 \dots w_k w_{k+1} \dots w_n. \quad (2.3)$$

The order of the Coxeter element is  $h$ , the Coxeter number.

Finally, root vectors  $\phi_i$  ( $i = 1, \dots, n$ ) are selected as representatives of the Weyl orbits as follows:

$$\phi_i = w_n w_{n-1} \dots w_{i+1}(\alpha_i). \quad (2.4)$$

Then, the image of each  $\phi_i$  ( $i = 1, \dots, n$ ) under the inverse Coxeter element is a positive root and successive images remain positive until they all change sign, remaining negative subsequently for the rest of the orbit.

With this machinery, the hypothesised expression of the exact  $S$ -matrix for simply-laced  $ADE$  affine Toda field theory is written in the following form:

$$S_{ab}(\theta) = \prod_{p=0}^{h-1} \{2p+1 + \epsilon_{ab}\}^{1/2(\lambda_a \cdot w^{-p} \phi_b)}, \quad (2.5)$$

where

$$\{x\} = \frac{(x-1)(x+1)}{(x-1+2B)(x+1-2B)}, \quad (x) = \frac{sh(\theta/2 + i\pi x/2h)}{sh(\theta/2 - i\pi x/2h)}. \quad (2.6)$$

$\theta = \theta_a - \theta_b$  is the difference of the rapidities and  $\lambda_a$  are dual vectors such that  $(\lambda_i \cdot \alpha_j) = \delta_{ij}$ . The coupling dependence enters through the universal function  $B(\beta) = \beta^2/(\beta^2 + 4\pi)$ .  $\epsilon_{ab}$  is defined as follows depending on the colour of the pair  $a, b$ :

$$\epsilon_{\bullet\bullet} = \epsilon_{\circ\circ} = 0, \quad \epsilon_{\circ\bullet} = -\epsilon_{\bullet\circ} = 1. \quad (2.7)$$

### 3. Boundary bootstrap

It is usually supposed that particles on a half line ( $-\infty < x \leq 0$ ) scatter as if the boundary were absent. In other words, scattering of particles on a half line can be described by the same  $S$ -matrix of the system defined on a whole line.

In the algebraic approach, the boundary reflection matrix is defined in terms of Zamolodchikov-Faddeev algebra[2]:

$$A_a^\dagger(\theta)B = K_a(\theta)A_a^\dagger(-\theta)B, \quad (3.1)$$

where  $A_a^\dagger$  is the creation operator of particle  $a$  and  $B$  is the boundary creation operator. Consistency requirements of the boundary reflection with the scattering and the three-point vertex function lead to some algebraic relations which impose stringent constraints to the solution of the boundary reflection matrix.

To begin with, the general unitarity requirement leads to the boundary unitarity relation:

$$K_a(\theta)K_a(-\theta) = 1. \quad (3.2)$$

Consistency requirement on the two particle process yields a constraint which is called the boundary Yang-Baxter equation:

$$K_b(\theta_b)S_{ab}(\theta_a + \theta_b)K_a(\theta_a)S_{ab}(\theta_a - \theta_b) = S_{ab}(\theta_a - \theta_b)K_a(\theta_a)S_{ab}(\theta_a + \theta_b)K_b(\theta_b). \quad (3.3)$$

When types of particles do not change during the reflection process as in the case of ATFT, eq.(3.3) is automatically satisfied. Crossing-unitarity relation is

$$K_a(\theta)K_{\bar{a}}(i\pi + \theta) = S_{aa}(2\theta). \quad (3.4)$$

This relation is non-linear in  $K$ , which is effective particularly in solving the scalar function of non-diagonal boundary reflection matrix.

Consistency requirement of the boundary reflection with the three-point vertex function leads to the boundary bootstrap equation:

$$K_c(\theta)(-if_c^{ab}) = (-if_c^{ab})K_a(\theta + i\bar{\theta}_{ac}^b)S_{ab}(2\theta + i\bar{\theta}_{ac}^b - i\bar{\theta}_{bc}^a)K_b(\theta - i\bar{\theta}_{bc}^a). \quad (3.5)$$

$f_c^{ab}$  is the three-point vertex function. The fusing angles  $\theta_{ab}^c$  is defined as  $m_c^2 = m_a^2 + m_b^2 - 2m_a m_b \cos \theta_{ab}^c$  and  $\bar{\theta} = i\pi - \theta$ . eq.(3.5) is represented pictorially in Fig. 1. For diagonal reflection, only the crossing-unitarity relation and the boundary bootstrap equation produce non-trivial constraints for the boundary reflection matrix.

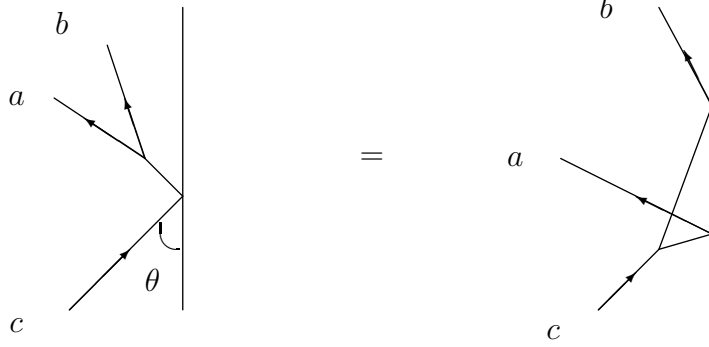


Figure 1.

The above boundary bootstrap equation involves the  $S$ -matrix in a rather non-trivial way. This fact makes it difficult to see the general structure of it. In order to separate the  $S$ -matrix from the boundary bootstrap equation, let the argument  $\theta$  be shifted by an amount of  $i\pi$  and take the conjugation of the particle indices. Then, it yields

$$K_{\bar{c}}(\theta + i\pi) = K_{\bar{a}}(\theta + i\pi + i\bar{\theta}_{ac}^b) S_{ab}(2\theta + i\bar{\theta}_{ac}^b - i\bar{\theta}_{bc}^a) K_{\bar{b}}(\theta + i\pi - i\bar{\theta}_{bc}^a). \quad (3.6)$$

Here the facts that  $S(\theta)$  is  $2\pi i$ -periodic and  $S_{\bar{a}\bar{b}}(\theta) = S_{ab}(\theta)$  are used. Fusing angle  $\theta_{ab}^c$  does not change under the conjugation.

Now, it seems natural to introduce a new function  $J_a(\theta)$ :

$$J_a(\theta) = \sqrt{K_a(\theta)/K_{\bar{a}}(i\pi + \theta)} = K_a(\theta)/\sqrt{S_{aa}(2\theta)}. \quad (3.7)$$

The second equality follows from the crossing-unitarity relation. On replacing the definition of  $J_a(\theta)$  into eq.(3.5) divided by eq.(3.6), one gets a very simple equation for the boundary bootstrap:

$$J_c(\theta) = J_a(\theta + i\bar{\theta}_{ac}^b) J_b(\theta - i\bar{\theta}_{bc}^a). \quad (3.8)$$

This equation may be depicted as in Fig. 2. At first glance, this equation seems to have nothing to do with the boundary conditions. But, in fact it does have something! For instance, in order to solve eq.(3.8) one needs to know the limiting behavior of  $J_a(\theta)$  at the strong and weak couplings, which depends on a given boundary potential.

There is one interesting coincidence. Namely, the same form of the equation as eq.(3.8) already appeared in ref.[6], where  $J_a(\theta)$  is interpreted as the classical limit (where  $S(\theta)$  tends to unity) of the exact  $K_a(\theta)$ . In general, the classical limit of  $K_a(\theta)$  need not be unity, as in cases with integrable non-trivial boundary potentials. For the present case of the Neumann boundary condition,  $J_a(\theta)$  also tends to unity as  $\beta \rightarrow 0$ .

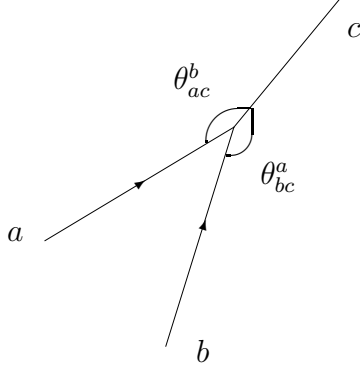


Figure 2.

In terms of  $J_a(\theta)$ , the unitarity relation eq.(3.2) changes into

$$J_a(\theta)J_a(-\theta) = 1, \quad (3.9)$$

and the crossing-unitarity relation eq.(3.4) simplifies to

$$J_a(\theta)J_{\bar{a}}(i\pi + \theta) = 1. \quad (3.10)$$

With the known conjecture[10] of the boundary reflection matrix for simply-laced *ADE* affine Toda field theory defined on a half line with the Neumann boundary condition, it is not a difficult observation to derive the following hypothesis:

$$J_b(\theta) = \prod_{p=0}^{h-1} [2p + 1/2 + \epsilon_b]^{1/2 \sum_a (\lambda_a \cdot w^{-p} \phi_b)}, \quad (3.11)$$

where

$$[x] = \frac{(x - 1/2)(x + 1/2)}{(x - 1/2 + B)(x + 1/2 - B)}. \quad (3.12)$$

$\theta$  is the rapidity and  $\epsilon_b$  is defined as follows depending on the colour of  $b$ :

$$\epsilon_{\bullet} = 1, \quad \epsilon_{\circ} = 0. \quad (3.13)$$

For reader's reference, some identities are listed below:

$$\{x\}_{2\theta} = [x/2]_{\theta}/[h - x/2]_{\theta}, \quad [2h + x] = [x], \quad [-x] = 1/[x]. \quad (3.14)$$

The above formula for the  $J$ -matrix in eq.(3.11) has a very similar dependence on the Coxeter element as the one for the  $S$ -matrix in eq.(2.5) and can be shown to satisfy all the necessary algebraic constraints quite analogously as in ref.[11] with minor modifications.

## 4. Conclusions

For purely elastic boundary reflection, the principle of boundary bootstrap plays a significant role in the algebraic study on the exact boundary reflection matrix since the boundary Yang-Baxter equation becomes trivially satisfied.

The boundary bootstrap in its original form has not allowed an easy attack on its general structure[3]. In this letter, the new matrix  $J_a(\theta) = \sqrt{K_a(\theta)/K_{\bar{a}}(i\pi + \theta)}$  which renders the boundary bootstrap more tractable was introduced and then by analysing the known conjectures on the boundary reflection matrix with the new version of the boundary bootstrap, the hypothesised expression of the boundary reflection matrix for simply-laced *ADE* affine Toda field theory defined on a half line with the Neumann boundary condition was obtained in terms of root systems.

Boundary conditions which are compatible with classical or quantum integrability has been investigated for ATFTs associated with simply-laced Lie algebras as well as non-simply-laced Lie algebras[2, 5, 6, 14, 15, 16]. Classical boundary reflection matrices corresponding to the various choices of the integrable boundary condition have been constructed by linearising the equation of motion around a background solution in refs.[5, 6], where some conjectures on the exact boundary reflection matrices have been also made.

Further studies on the new version of the boundary bootstrap would be useful for finding exact boundary reflection matrices corresponding to integrable non-trivial boundary potentials. To this end, one should take into account renormalisation of the boundary parameters[15, 16] which is related to the limiting behavior of boundary reflection matrix at the strong and weak couplings and stability of the particle spectrum[17].

It is also hoped that deeper understandings in boundary reflection matrix will provide a new sort of insights into the unified formulation of ATFTs associated with simply-laced Lie algebras as well as non-simply-laced Lie algebras. Boundary reflection matrices for some non-simply-laced ATFTs have been obtained in refs.[4, 18].

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